

A MATRIX WEIGHTED T_1 THEOREM FOR MATRIX KERNELLED CALDERÓN-ZYGMUND OPERATORS - I

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ABSTRACT. In this series of two papers, we will prove a natural matrix weighted T_1 theorem for matrix kernelled CZOs. In the current paper, we will prove matrix weighted norm inequalities for matrix symbolled paraproducts via a general matrix weighted Carleson embedding theorem. Along the way, we will also provide a stopping time proof of the identification of $L^p(W)$ as a weighted Triebel-Lizorkin space when W is a matrix A_p weight.

1. INTRODUCTION

Weighted norm inequalities for Calderón-Zygmund operators (or CZOs for short) acting on ordinary $L^p(\mathbb{R}^d)$ is a classical topic that goes back to the 1970's with the seminal works [7, 9]. On the other hand, it is well known that proving matrix weighted norm inequalities for CZOs is a very difficult task, and because of this, matrix weighted norm inequalities for certain CZOs have only recently been investigated (see [21, 22] for specific details of these difficulties). In particular, if n and d are natural numbers and if $W : \mathbb{R}^d \rightarrow \mathcal{M}_n(\mathbb{C})$ is positive definite a. e. (where as usual $\mathcal{M}_n(\mathbb{C})$ is the algebra of $n \times n$ matrices with complex scalar entries), then define $L^p(W)$ for $1 < p < \infty$ to be the space of measurable functions $\vec{f} : \mathbb{R}^d \rightarrow \mathbb{C}^n$ with norm

$$\|\vec{f}\|_{L^p(W)}^p = \int_{\mathbb{R}^d} |W^{\frac{1}{p}}(x)\vec{f}(x)|^p dx.$$

It was proved by F. Nazarov and S. Treil, M. Goldberg, and A. Volberg, respectively in [8, 14, 22] that certain CZOs are bounded on

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$L^p(W)$ when $1 < p < \infty$ if W is a matrix A_p weight, which means that

$$\sup_{\substack{I \subset \mathbb{R}^d \\ I \text{ is a cube}}} \frac{1}{|I|} \int_I \left(\frac{1}{|I|} \int_I \|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(t)\|^{p'} dt \right)^{\frac{p}{p'}} dx < \infty \quad (1.1)$$

where p' is the conjugate exponent of p (note that an operator T acting on scalar functions can be canonically extended to \mathbb{C}^n valued functions via the action of T on its coordinate functions.)

Now note that CZOs with matrix valued kernels acting on \mathbb{C}^n valued functions appear very naturally in various branches of mathematics (and as a particular example see [10] for extensive applications of matrix kernelled CZOs to geometric function theory.) Despite this and the fact that the theory of matrix weights has numerous applications to Toeplitz operators, multivariate prediction theory, and even to the study of finitely generated shift invariant subspaces of unweighted $L^p(\mathbb{R}^d)$ (see [14, 16, 22]), virtually nothing is known regarding matrix weighted norm inequalities for matrix kernelled CZOs or related operators.

The purpose of this series of two papers is therefore to investigate the boundedness of matrix kernelled CZOs on $L^p(W)$ when W is a matrix A_p weight. We will need to introduce some more notation before we state our main result. It is well known (see [8] for example) that for a matrix weight W , a cube I , and any $1 < p < \infty$, there exists positive definite matrices V_I and V'_I such that $|I|^{-\frac{1}{p}} \|\chi_I W^{\frac{1}{p}} \vec{e}\|_{L^p} \approx |V_I \vec{e}|$ and $|I|^{-\frac{1}{p'}} \|\chi_I W^{-\frac{1}{p}} \vec{e}\|_{L^{p'}} \approx |V'_I \vec{e}|$ for any $\vec{e} \in \mathbb{C}^n$, where $\|\cdot\|_{L^p}$ is the canonical $L^p(\mathbb{R}^d; \mathbb{C}^n)$ norm and the notation $A \approx B$ as usual means that two quantities A and B are bounded above and below by a constant multiple of each other. Note that it is easy to see that $\|V_I V'_I\| \geq 1$ for any cube I . We will say that W is a matrix A_p weight if the product $V_I V'_I$ has uniformly bounded matrix norm with respect to all cubes $I \subset \mathbb{R}^d$ (note that this condition is easily seen to be equivalent to (1.1).) Also note that when $p = 2$ we have $V_I = (m_I W)^{\frac{1}{2}}$ and $V'_I = (m_I(W^{-1}))^{\frac{1}{2}}$ where $m_I W$ is the average of W on I , so that the matrix A_2 condition takes on a particularly simple form that is very similar to the scalar A_2 condition.

Now let $T : L^2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L^2(\mathbb{R}^d; \mathbb{C}^n)$ be a densely defined operator where the dense domain contains at least the indicator function of all cubes. If $1 < p < \infty$ and W is a matrix A_p weight, then we will call T

a “ W -weighted CZO” with associated matrix kernel $K : \mathbb{R}^d \times \mathbb{R}^d \setminus \Delta \rightarrow \mathcal{M}_n(\mathbb{C})$ (where as usual $\Delta \subset \mathbb{R}^d \times \mathbb{R}^d$ is the diagonal) if the following three conditions are true: first,

$$T\vec{f}(x) = \int_{\mathbb{R}^d} K(x, y)\vec{f}(y) dy, \quad x \notin \text{supp}(f)$$

for all \vec{f} in the dense domain of T with compact support. Second, for each cube $I \subset \mathbb{R}^d$, assume that the matrix function $V_I K(x, y)V_I^{-1}$ satisfies the “standard kernel estimates”

$$|V_I K(x, y)V_I^{-1}| \leq \frac{C}{|x - y|^d},$$

$$|V_I(K(x, y) - K(x', y))V_I^{-1}| + |V_I(K(y, x) - K(y, x'))V_I^{-1}| \leq C \frac{|x - x'|^\delta}{|x - y|^{d+\delta}}$$

for all $x, x', y \in \mathbb{R}^d$ with $|x - y| > 2|x - x'|$ where $\delta, C > 0$ are independent of I . Third, assume that T satisfies the “weak boundedness property”

$$\sup_{\substack{I \subset \mathbb{R}^d \\ I \text{ is a cube}}} \frac{1}{|I|} \|\langle T(1_I), 1_I \rangle_{L^2}\| < \infty$$

where 1_I is the indicator function of the cube I and $\|\cdot\|$ is any matrix norm on $\mathcal{M}_n(\mathbb{C})$. Moreover, if $1 < p < \infty$ and W is a matrix A_p weight, then let BMO_W^p be the space of locally integrable functions $B : \mathbb{R}^d \rightarrow \mathcal{M}_n(\mathbb{C})$ where

$$\left\{ \begin{array}{ll} \sup_{\substack{I \subset \mathbb{R}^d \\ I \text{ is a cube}}} \frac{1}{|I|} \int_I \|W^{\frac{1}{p}}(x)(B(x) - m_I B)V_I^{-1}\|^p dx < \infty & : \text{ if } 2 \leq p < \infty \\ \sup_{\substack{I \subset \mathbb{R}^d \\ I \text{ is a cube}}} \frac{1}{|I|} \int_I \|W^{-\frac{1}{p}}(x)(B^*(x) - m_I B^*)(V_I')^{-1}\|^{p'} dx < \infty & : \text{ if } 1 < p \leq 2 \end{array} \right\}.$$

Our main goal in these two papers will be to prove the following theorem

Theorem 1.1. *Let $1 < p < \infty$. If W is a matrix A_p weight and T is a W -weighted CZO, then T is bounded on $L^p(W)$ if and only if $T1 \in BMO_W^p$ and $T^*1 \in BMO_{W^{1-p'}}^{p'}$.*

In this paper, however, we will focus our attention towards proving matrix weighted norm inequalities for dyadic paraproducts, which will be used to prove Theorem 1.1 in part II. In particular, let \mathcal{D} be a

dyadic system of cubes in \mathbb{R}^d and let $\{h_I^i\}_{I \in \mathcal{D}, i \in \{1, \dots, 2^d-1\}}$ be a system of Haar functions adapted to \mathcal{D} . Given a locally integrable function $B : \mathbb{R}^d \rightarrow \mathcal{M}_n(\mathbb{C})$, define the dyadic paraproduct π_B with respect to a dyadic grid \mathcal{D} by

$$\pi_B \vec{f} = \sum_{I \in \mathcal{D}} B_I(m_I \vec{f}) h_I \quad (1.2)$$

where B_I is the matrix of Haar coefficients of the entries of B with respect to I . In this paper we will prove the following theorem

Theorem 1.2. *Let $1 < p < \infty$. If W is a matrix A_p weight then π_B is bounded on $L^p(W)$ if and only if $B \in BMO_W^p$ (where here the supremum defining BMO_W^p is taken over all $I \in \mathcal{D}$ instead of all cubes I .)*

Let us comment that restricting oneself to W -weighted CZOs is in fact quite natural. In particular, note that Theorem 1.1 is *false* for general matrix A_p weights and matrix kernelled CZOs, and in the last section we will construct a very simple example, for each $1 < p < \infty$, of a matrix A_p weight W and a matrix kernelled CZO T with $T1 = T^*1 = 0$ but where T is *not* bounded on $L^p(W)$.

Moreover, let $A = \{A_I\}_{I \in \mathcal{D}} \subset \mathcal{M}_n(\mathbb{C})$ be a sequence of matrices. We will then prove (see Section 4) that given a matrix A_p weight W , the Haar multiplier

$$\vec{f} \mapsto \sum_{I \in \mathcal{D}} A_I \vec{f}_I h_I$$

is bounded on $L^p(W)$ if and only if $\sup_{I \in \mathcal{D}} \|V_I A_I V_I^{-1}\| < \infty$. On the other hand, in the last section we will exhibit a very simple example of a sequence A and a matrix A_p weight W , for each $1 < p < \infty$, where $\sup_{I \in \mathcal{D}} \|V_I A_I V_I^{-1}\| = \infty$. Similarly in the last section we will construct a matrix function $B \in BMO$ (the ordinary John-Nirenberg BMO space) and a matrix A_p weight W for each $1 < p < \infty$ where $B \notin BMO_W^p$.

The proof of Theorem 1.2 will require the following matrix weighted Carleson embedding theorem, which is obviously of independent interest itself.

Theorem 1.3. *Let $1 < p < \infty$. If W is a matrix A_p weight and $A := \{A_I\}_{I \in \mathcal{D}}$ is a sequence of matrices, then the following are equivalent:*

(a) The operator Π_A defined by

$$\Pi_A \vec{f} := \sum_{I \in \mathcal{D}} V_I A_I m_I (W^{-\frac{1}{p}} \vec{f}) h_I$$

is bounded on $L^p(\mathbb{R}^d; \mathbb{C}^n)$

(b)

$$\sup_{J \in \mathcal{D}} \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \|V_I A_I V_I^{-1}\|^2 < \infty$$

(c) There exists $C > 0$ independent of $J \in \mathcal{D}$ such that

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} A_I^* V_I^2 A_I < C V_J^2$$

if $2 \leq p < \infty$, and

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} A_I (V_I')^2 A_I^* < C (V_J')^2$$

if $1 < p \leq 2$.

Furthermore, the operator norm in (a) and the supremums in (b) and (c) are equivalent in the sense that they are independent of the sequence A . Finally, a matrix function $B \in BMO_W^p$ if and only if the sequence of Haar coefficients of B satisfies any of the above equivalent conditions.

Despite the perhaps strange appearance of Theorem 1.3, first note that if w is a scalar A_p weight, then clearly a locally integrable scalar function b is in BMO_W^p if and only if b is in BMO (i. e. the classical John-Nirenberg BMO space), which in this case is also equivalent to

$$\sup_{\substack{I \subset \mathbb{R}^d \\ I \text{ is a cube}}} \frac{1}{w(I)} \int_I w(x) |b(x) - m_I b| dx < \infty \quad (1.3)$$

where $w(I) = \int_I w(x) dx$. In fact, it is well known (and easy to prove) that if w is a scalar A_∞ weight, then b satisfies (1.3) if and only if $b \in BMO$ (see [13] for details.)

Furthermore, note that when $p = 2$, the implication (c) \Rightarrow (a) in Theorem 1.3 gives us that (after replacing \vec{f} with $W^{\frac{1}{2}} \vec{f}$ and replacing A_I with $(m_I W)^{-\frac{1}{2}} A_I$)

$$\sum_{I \in \mathcal{D}} |A_I(m_I \vec{f})|^2 \lesssim C \|\vec{f}\|_{L^2(w)}^2$$

whenever W is a matrix A_2 weight and $\{A_I\}_{I \in \mathcal{D}}$ is a “ W -Carleson sequence” of matrices in the sense that

$$\sum_{I \in \mathcal{D}(J)} A_I^* A_I < C \int_J W(x) dx$$

holds for all $J \in \mathcal{D}$.

Interestingly, note that this “weighted Carleson embedding Theorem” in the scalar $p = 2$ setting appears as Lemma 5.7 in [17] for scalar A_∞ weights and was implicitly used in sharp form by O. Beznosova [2] (see (2.4) and (2.5) in [2]) to prove sharp weighted norm inequalities for scalar paraproducts. We will further discuss the connection between Theorem 1.3 and sharp matrix weighted norm inequalities for scalar kernelled CZOs in the final section.

The main tool for proving Theorem 1.3 will be an adaption of the stopping time arguments from [11, 19] to the matrix weighted $p \neq 2$ setting. Moreover, the proof will require the identification from [14, 22] of $L^p(W)$ as a weighted Triebel-Lizorkin space for $1 < p < \infty$ when W is a matrix A_p weight, which more precisely says that

$$\|\vec{f}\|_{L^p(W)}^p \approx \int_{\mathbb{R}^d} \left(\sum_{I \in \mathcal{D}} \frac{|V_I \vec{f}_I|^2}{|I|} \chi_I(x) \right)^{\frac{p}{2}} dx \quad (1.4)$$

where \vec{f}_I is the Haar coefficient of \vec{f} . Note that we will use our stopping time to give a new and more classical stopping time proof of (1.4), which could be thought of as the third contribution of the current paper (and which provides a simpler approach when compared to the ones in [14, 22].) It is hoped that our proofs of Theorem 1.3 and (1.4) will convince the reader of the overall usefulness of our stopping approach to matrix weighted norm inequalities and will generate interest in extending other stopping time arguments to the matrix weighted setting (such as the ones pioneered by M. Lacey, S. Petermichl, and M. C. Reguera in [12], which will be discussed further in the last section.)

It is also hoped that the results in this series of two papers will convince the reader of the following philosophy: what is true in the scalar A_p /scalar CZO setting should largely be true in the matrix setting after one takes noncommutativity into account.

We will end this introduction by outlining the contents of each section. In the next section we will extend the stopping time arguments in [11, 19] to the $p \neq 2$ matrix weighted setting and show that this stopping

time is a decaying stopping time in the sense of [11], which will then be used to prove (1.4). In the third section, we will prove Theorems 1.2 and 1.3 by utilizing (1.4) in conjunction with our stopping time arguments. Finally in Section four, we will construct the examples discussed earlier in this introduction. Moreover, we will present some interesting open problems, including other related matrix weighted BMO spaces and their possible equivalences to BMO_W^p .

2. WEIGHTED HAAR MULTIPLIERS AND STOPPING TIMES

We will now describe the Haar multipliers and the stopping time that will be needed throughout this paper. Define the constant Haar multiplier $M_{W,p}$ by

$$M_{W,p}\vec{f} := \sum_{I \in \mathcal{D}} V_I \vec{f}_I h_I.$$

Note that trivially π_B is bounded on $L^p(W)$ if and only if $W^{\frac{1}{p}}\pi_B W^{-\frac{1}{p}}$ is bounded on $L^p(\mathbb{R}^d; \mathbb{C}^n)$, and note that

$$W^{\frac{1}{p}}\pi_B W^{-\frac{1}{p}} = W^{\frac{1}{p}}(M_{W,p})^{-1} \left(M_{W,p}\pi_B W^{-\frac{1}{p}} \right).$$

The main goal of this section will be to extend the stopping time arguments in [11, 19] to the matrix weighted setting and then use these arguments to prove that $W^{\frac{1}{p}}(M_{W,p})^{-1}$ is bounded and invertible on $L^p(\mathbb{R}^d; \mathbb{C}^n)$ if W is a matrix A_p weight, so that one only needs to deal with $M_{W,p}\pi_B W^{-\frac{1}{p}}$ in order to prove Theorem 1.2. Furthermore, note that dyadic Littlewood-Paley theory immediately says that the boundedness and invertibility of $W^{\frac{1}{p}}(M_{W,p})^{-1}$ is equivalent to (1.4).

Now assume that W is a matrix A_p weight. For any cube $I \in \mathcal{D}$, let $\mathcal{J}(I)$ be the collection of maximal $J \in \mathcal{D}(I)$ such that

$$\|V_J V_I^{-1}\|^p > \lambda_1 \quad \text{or} \quad \|V_J^{-1} V_I\|^{p'} > \lambda_2 \quad (2.1)$$

for some $\lambda_1, \lambda_2 > 1$ to be specified later. Also, let $\mathcal{F}(I)$ be the collection of dyadic subcubes of I not contained in any cube $J \in \mathcal{J}(I)$, so that clearly $J \in \mathcal{F}(J)$ for any $J \in \mathcal{D}(I)$.

Let $\mathcal{J}^0(I) := \{I\}$ and inductively define $\mathcal{J}^j(I)$ and $\mathcal{F}^j(I)$ for $j \geq 1$ by $\mathcal{J}^j(I) := \bigcup_{J \in \mathcal{J}^{j-1}(I)} \mathcal{J}(J)$ and $\mathcal{F}^j(I) := \bigcup_{J \in \mathcal{J}^{j-1}(I)} \mathcal{F}(J)$. Clearly the cubes in $\mathcal{J}^j(I)$ for $j > 0$ are pairwise disjoint. Furthermore, since $J \in \mathcal{F}(J)$ for any $J \in \mathcal{D}(I)$, we have that $\mathcal{D}(I) = \bigcup_{j=0}^{\infty} \mathcal{F}^j(I)$. We will slightly abuse notation and write $\bigcup \mathcal{J}(I)$ for the

set $\bigcup_{J \in \mathcal{J}(I)} J$ and write $|\bigcup \mathcal{J}(I)|$ for $|\bigcup_{J \in \mathcal{J}(I)} J|$. We will now show that \mathcal{J} is a decaying stopping time in the sense of [11].

Lemma 2.1. *Let $1 < p < \infty$ and let W be a matrix A_p weight. For $\lambda_1, \lambda_2 > 1$ large enough, we have that $|\bigcup \mathcal{J}^j(I)| \leq 2^{-j}|I|$ for every $I \in \mathcal{D}$.*

Proof. By iteration, it is enough to prove the lemma for $j = 1$. For $I \in \mathcal{D}$, let $\mathcal{G}(I)$ denote the collection of maximal $J \in \mathcal{D}(I)$ such that the first inequality (but not necessarily the second inequality) in (2.1) holds. Then by maximality and elementary linear algebra, we have that

$$\left| \bigcup_{J \in \mathcal{G}(I)} J \right| = \sum_{J \in \mathcal{G}(I)} |J| \lesssim \frac{1}{\lambda_1} \sum_{J \in \mathcal{G}(I)} \int_J \|W^{\frac{1}{p}}(y) V_I^{-1}\|^p dy \leq \frac{C_1 |I|}{\lambda_1}$$

for some $C_1 > 0$ only depending on n and d .

On the other hand, let For $I \in \mathcal{D}$, let $\tilde{\mathcal{G}}(I)$ denote the collection of maximal $J \in \mathcal{D}(I)$ such that the second inequality (but not necessarily the first inequality) in (2.1) holds. Then by the matrix A_p condition we have

$$\left| \bigcup_{J \in \tilde{\mathcal{G}}(I)} J \right| \leq \frac{C_2}{\lambda_2} \sum_{J \in \tilde{\mathcal{G}}(I)} \int_J \|W^{-\frac{1}{p}}(y) V_I\|^{p'} dy \leq \frac{C'_2 \|W\|_{A_p}^{\frac{p'}{p}}}{\lambda_2} |I|$$

for some C'_2 only depending on n and d . The proof is now completed by setting $\lambda_1 = 4C_1$ and $\lambda_2 = 4C'_2 \|W\|_{A_p}^{\frac{p'}{p}}$. \square

While we will not have a need to discuss matrix $A_{p,\infty}$ weights in detail in this paper, note that in fact Lemma 3.1 in [22] immediately gives us that Lemma 2.1 holds for matrix $A_{p,\infty}$ weights (with possibly larger λ_2 of course.)

The next main result will be an “ L^p Cotlar-Stein lemma” (Lemma 2.2) that is a vector version of Lemma 8 in [11]. We will need a few preliminary definitions before we state this result. Fix $J_0 \in \mathcal{D}$ with side-length 1 and with $0 \in J_0$ and let $\mathcal{J}^j := \mathcal{J}^j(J_0)$ and $\mathcal{F}^j := \mathcal{F}^j(J_0)$. Now for each $j \in \mathbb{N}$ let Δ_j be defined by

$$\Delta_j \vec{f} := \sum_{I \in \mathcal{F}^j} \vec{f}_I h_I,$$

and write $\vec{f}_j := \Delta_j \vec{f}$.

Lemma 2.2. *Let the \mathcal{F}^j 's be as above and write $T_j := T\Delta_j$ for any linear operator T acting on \mathbb{C}^n valued functions defined on \mathbb{R}^d . If $T = \sum_{j=1}^{\infty} T_j$, and if there exists $C > 0$ and $0 < c < 1$ such that*

$$\int_{\mathbb{R}^d} |T_j \vec{f}|^{\frac{p}{2}} |T_k \vec{f}|^{\frac{p}{2}} dx \lesssim c^{|j-k|} \|\vec{f}_j\|_{L^p}^{\frac{p}{2}} \|\vec{f}_k\|_{L^p}^{\frac{p}{2}}$$

for every $j, k \in \mathbb{N}$, then T is bounded on $L^p(\mathbb{R}^d; \mathbb{C}^n)$.

Proof. It follows directly from Lemma 7 in [11] and elementary linear algebra that

$$\sum_{j=1}^{\infty} \|\vec{f}_j\|_{L^p}^p \lesssim \|\vec{f}\|_{L^p}^p$$

whenever $f \in L^p(\mathbb{R}^d; \mathbb{C}^n)$. The proof of Lemma 2.2 is now identical to the proof of Lemma 8 in [11]. \square

Theorem 2.3. *Let $1 < p < \infty$. If W is a matrix A_p weight, then $W^{\frac{1}{p}} M_{W,p}^{-1}$ is bounded on $L^p(\mathbb{R}^d; \mathbb{C}^n)$.*

Proof. Obviously it is enough to prove that the operator T defined by

$$T\vec{f} := \sum_{I \in \mathcal{D}(J_0)} W^{\frac{1}{p}} V_I^{-1} \vec{f}_I h_I$$

is bounded on $L^p(\mathbb{R}^d; \mathbb{C}^n)$. Note that we also clearly have $T = \sum_{j=1}^{\infty} T_j$.

For each $I \in \mathcal{D}$, let

$$M_I \vec{f} := \sum_{J \in \mathcal{F}(I)} V_J^{-1} \vec{f}_J h_J$$

so that

$$T_j \vec{f} = \sum_{I \in \mathcal{J}^{j-1}} W^{\frac{1}{p}} M_I \vec{f}.$$

Since $V_I M_I$ is a constant Haar multiplier and since $\|V_I V_J^{-1}\|^p \leq \|W\|_{A_p}$ if $J \in \mathcal{F}(I)$, we immediately have that

$$\|V_I M_I \vec{f}\|_{L^p}^p \lesssim \|W\|_{A_p} \|\vec{f}\|_{L^p}^p.$$

Now we will show that each T_j is bounded. To that end, we have that

$$\begin{aligned} \int_{\mathbb{R}^d} |T_j f|^p dx &= \int_{\bigcup_{\mathcal{J}^{j-1}} \setminus \bigcup_{\mathcal{J}^j}} |T_j f|^p dx + \int_{\bigcup_{\mathcal{J}^j}} |T_j f|^p dx \\ &:= (A) + (B). \end{aligned}$$

Since $\|W^{\frac{1}{p}}(x)V_J^{-1}\|^p \lesssim 1$ on $J \setminus \bigcup \mathscr{J}(J)$, we can estimate (A) first as follows:

$$\begin{aligned}
(A) &= \sum_{J \in \mathscr{J}^{j-1}} \int_{J \setminus \bigcup \mathscr{J}(J)} |T_j \vec{f}|^p dx \\
&= \sum_{J \in \mathscr{J}^{j-1}} \int_{J \setminus \bigcup \mathscr{J}(J)} |W^{\frac{1}{p}}(x) M_J \vec{f}(x)|^p dx \\
&\leq \sum_{J \in \mathscr{J}^{j-1}} \int_{J \setminus \bigcup \mathscr{J}(J)} \|W^{\frac{1}{p}}(x) V_J^{-1}\|^p |V_J M_J \vec{f}(x)|^p dx \\
&\lesssim \sum_{J \in \mathscr{J}^{j-1}} \int_J |V_J M_J \vec{f}|^p dx \\
&\lesssim \|W\|_{A_p} \|\vec{f}_j\|_{L^p}^p \\
&\lesssim \|W\|_{A_p} \|\vec{f}\|_{L^p}^p.
\end{aligned}$$

As for (B), note that $M_J \vec{f}$ is constant on $I \in \mathscr{J}(J)$, and so we will refer to this constant by $M_J \vec{f}(I)$. We then estimate (B) as follows:

$$\begin{aligned}
(B) &= \int_{\bigcup \mathscr{J}^j} |T_j \vec{f}|^p dx \\
&\leq \sum_{J \in \mathscr{J}^{j-1}} \sum_{I \in \mathscr{J}(J)} \int_I |W^{\frac{1}{p}}(x) M_J \vec{f}|^p dx \\
&\leq \sum_{J \in \mathscr{J}^{j-1}} \sum_{I \in \mathscr{J}(J)} |I| |V_J M_J \vec{f}(I)| \left(\frac{1}{|I|} \int_I \|W^{\frac{1}{p}}(x) V_J^{-1}\|^p dx \right) \\
&\lesssim \sum_{J \in \mathscr{J}^{j-1}} \sum_{I \in \mathscr{J}(J)} |I| |V_J M_J \vec{f}(I)| \\
&= \sum_{J \in \mathscr{J}^{j-1}} \sum_{I \in \mathscr{J}(J)} \int_I |V_J M_J \vec{f}(x)|^p dx \\
&\lesssim \|W\|_{A_p} \|\vec{f}_j\|_{L^p}^p \\
&\lesssim \|W\|_{A_p} \|\vec{f}\|_{L^p}^p. \tag{2.2}
\end{aligned}$$

To finish the proof, we claim that there exists $0 < c < 1$ such that

$$\int_{\bigcup \mathscr{J}^{k-1}} |T_j \vec{f}|^p dx \lesssim c^{k-j} \|\vec{f}_j\|_{L^p}^p$$

whenever $k > j$. If we define $M_j \vec{f}$ as

$$M_j \vec{f} := \sum_{I \in \mathcal{J}^{j-1}} M_I \vec{f},$$

then $M_j \vec{f}$ is constant on $J \in \mathcal{J}^j$. Thus, we have that

$$\begin{aligned} \int_{\bigcup \mathcal{J}^{k-1}} |T_j \vec{f}|^p dx &= \sum_{J \in \mathcal{J}^j} \sum_{I \in \mathcal{J}^{k-j-1}(J)} \int_I |W^{\frac{1}{p}}(x) M_j \vec{f}(J)|^p dx \\ &\leq \sum_{J \in \mathcal{J}^j} \sum_{I \in \mathcal{J}^{k-j-1}(J)} |J| |V_J M_j \vec{f}(J)|^p \left(\frac{1}{|J|} \int_I \|W^{\frac{1}{p}}(x) V_J^{-1}\|^p dx \right). \end{aligned}$$

However,

$$\begin{aligned} |J| |V_J M_j \vec{f}(J)|^p &\lesssim \int_J |W^{\frac{1}{p}}(x) M_j \vec{f}(J)|^p dx \\ &= \int_J |T_j \vec{f}(x)|^p dx. \end{aligned} \quad (2.3)$$

On the other hand, it is not hard to show that $|W^{\frac{1}{p}}(x) \vec{e}|^p$ is a scalar A_p weight for any $\vec{e} \in \mathbb{C}^n$ (see [8]), which by the classical reverse Hölder inequality means that we can pick some $q > p$ and use Hölder's inequality in conjunction with Lemma 2.1 to get

$$\begin{aligned} \frac{1}{|J|} \sum_{I \in \mathcal{J}^{k-j-1}(J)} \int_I \|W^{\frac{1}{p}}(x) V_J^{-1}\|^p dx &= \frac{1}{|J|} \int_{\bigcup \mathcal{J}^{k-j-1}(J)} \|W^{\frac{1}{p}}(x) V_J^{-1}\|^p dx \\ &\leq \frac{1}{|J|} \left(\int_{\bigcup \mathcal{J}^{k-j-1}(J)} \|W^{\frac{1}{p}}(x) V_J^{-1}\|^q dx \right)^{\frac{p}{q}} (2^{-(k-j-1)} |J|)^{1-\frac{p}{q}} \\ &\lesssim 2^{-(k-j-1)(1-\frac{p}{q})} \left(\frac{1}{|J|} \int_J \|W^{\frac{1}{p}}(x) V_J^{-1}\|^q dx \right)^{\frac{p}{q}} \\ &\lesssim 2^{-(k-j-1)(1-\frac{p}{q})}. \end{aligned} \quad (2.4)$$

Combining (2.3) with (2.4), we get that

$$\begin{aligned} \int_{\bigcup \mathcal{J}^{k-1}} |T_j \vec{f}|^p dx &\lesssim 2^{-(k-j-1)(1-\frac{p}{q})} \int_{\bigcup \mathcal{J}^j} |T_j \vec{f}|^p dx \\ &\lesssim \|W\|_{A_p} 2^{-(k-j-1)(1-\frac{p}{q})} \|\vec{f}\|_{L^p}^p. \end{aligned}$$

Finally, note that this estimate combined with the Cauchy-Schwarz inequality and Lemma 2.2 completes the proof since $T_k \vec{f}$ is supported on $\bigcup \mathcal{J}^{k-1}$. \square

Note that the proof of Theorem 2.3 only requires Lemma 2.1 and the fact that $|W^{\frac{1}{p}}(x)\vec{e}|^p$ satisfies a reverse Hölder inequality for each \vec{e} . In particular, our proof (as do proofs in [14, 22]) holds for matrix A_p weights.

Furthermore, note that Theorem 2.3 easily gives us that $W^{\frac{1}{p}}M_{W,p}^{-1}$ is also invertible on $L^p(\mathbb{R}^d; \mathbb{C}^n)$ with bounded inverse when W is a matrix A_p weight. In particular, it is easy to see that W is a matrix A_p weight if and only if $W^{1-p'}$ is a matrix $A_{p'}$ weight. Thus, we have that $(W^{1-p'})^{\frac{1}{p'}} M_{W^{1-p'}, p'}^{-1} = W^{-\frac{1}{p}} M_{W^{1-p'}, p'}^{-1}$ is bounded on $L^{p'}(\mathbb{R}^d; \mathbb{C}^n)$ if W is a matrix A_p weight, so by duality we have that $M_{W^{1-p'}, p'}^{-1} W^{-\frac{1}{p}}$ is bounded on $L^p(\mathbb{R}^d; \mathbb{C}^n)$. However, one can check very easily that

$$M_{W^{1-p'}, p'} \vec{f} = \sum_{I \in \mathcal{D}} V_I' \vec{f}_I h_I,$$

which by the matrix A_p condition means that $M_{W,p} M_{W^{1-p'}, p'}$ is a bounded Haar multiplier. We can then finally conclude that

$$\left(W^{\frac{1}{p}} M_{W,p}^{-1} \right)^{-1} = M_{W,p} W^{-\frac{1}{p}} = (M_{W,p} M_{W^{1-p'}, p'}) \left(M_{W^{1-p'}, p'}^{-1} W^{-\frac{1}{p}} \right)$$

is bounded on $L^p(\mathbb{R}^d; \mathbb{C}^n)$.

3. PROOF OF THEOREM 1.2

In this section we will prove Theorem 1.2. First we will need the following preliminary lemmas, the first of which is from [14].

Lemma 3.1. *Suppose that A is an $n \times n$ matrix where $|A\vec{e}| \geq |\vec{e}|$ for any $\vec{e} \in \mathbb{C}^n$. If $|\det A| \leq \delta$ for some $\delta \geq 0$, then $\|A\| \leq \delta$ where $\|\cdot\|$ is the canonical matrix norm on $\mathcal{M}_n(\mathbb{C})$.*

Proof. The proof follows from elementary linear algebra. \square

Lemma 3.2. *If W is a matrix A_p weight then*

$$|V_I' \vec{e}| \approx |m_I(W^{-\frac{1}{p}}) \vec{e}|$$

for any $\vec{e} \in \mathbb{C}^n$.

Proof. First we show that

$$\|V'_I \left(m_I(W^{-\frac{1}{p}}) \right)^{-1}\| \leq C$$

for some C independent of $I \in \mathcal{D}$, which will prove half of the lemma. Furthermore, note that the proof of this inequality will in fact also complete the other half of the proof. Since $W^{1-p'}$ is a matrix $A_{p'}$ weight, Proposition 2.2 in [22] says that W satisfies the “reverse matrix Jensen inequality”

$$\det V'_I \leq C \exp \left(m_I \log \det(W^{-\frac{1}{p}}) \right)$$

for any $I \in \mathcal{D}$ where C is independent of I . Combining this with the matrix Jensen inequality (Lemma 7.2 in [14]) we have that

$$\det V'_I \leq C \det m_I(W^{-\frac{1}{p}})$$

so that $\det V'_I(m_I(W^{-\frac{1}{p}})^{-1}) \leq C$.

Moreover, note that for any $\vec{e} \in \mathbb{C}^n$ we have

$$|m_I(W^{-\frac{1}{p}})\vec{e}| \leq \frac{1}{|I|} \int_I |W^{-\frac{1}{p}}(x)\vec{e}| dx \leq \left(\frac{1}{|I|} \int_I |W^{-\frac{1}{p}}(x)\vec{e}|^{p'} dx \right)^{\frac{1}{p'}} \leq |V'_I \vec{e}|$$

which means that

$$|V'_I(m_I(W^{-\frac{1}{p}})^{-1})\vec{e}| \geq |\vec{e}|$$

for any $\vec{e} \in \mathbb{C}^n$. The proof now follows immediately from Lemma 3.1. \square

Now note that by definition

$$(M_{W,p}\pi_B W^{-\frac{1}{p}})\vec{f} = \sum_{I \in \mathcal{D}} V_I B_I m_I(W^{-\frac{1}{p}}\vec{f})h_I.$$

Thus, since π_B is bounded on $L^p(W)$ if and only if $M_{W,p}\pi_B W^{-\frac{1}{p}}$ is bounded on $L^p(\mathbb{R}^d; \mathbb{C}^n)$, Theorem 1.2 immediately follows from Theorem 1.3, which we now prove.

Proof of Theorem 1.3: First note that if A_I is the sequence of Haar coefficients of some matrix valued function B , then (c) is equivalent to the original definition of BMO_W^p by an easy application of (1.4) and elementary linear algebra.

(b) \Rightarrow (a): By dyadic Littlewood-Paley theory, we need so show that

$$\int_{\mathbb{R}^d} \left(\sum_{I \in \mathcal{D}} \frac{|V_I A_I m_I(W^{-\frac{1}{p}}\vec{f})|^2}{|I|} \chi_I(t) \right)^{\frac{p}{2}} dt \lesssim \|\vec{f}\|_{L^p}^p$$

for any $f \in L^p(\mathbb{R}^d; \mathbb{C}^n)$. To that end, if $\epsilon > 0$ is small enough, then the reverse Hölder inequality gives us that

$$\begin{aligned}
|V_I A_I m_I(W^{-\frac{1}{p}} \vec{f})| &\leq \|V_I A_I V_I^{-1}\| |V_I m_I(W^{-\frac{1}{p}} \vec{f})| \\
&= \|V_I A_I V_I^{-1}\| \frac{1}{|I|} \left| \int_I V_I W^{-\frac{1}{p}}(y) \vec{f}(y) dy \right| \\
&\leq \|V_I A_I V_I^{-1}\| \left(\frac{1}{|I|} \int_I \|V_I W^{-\frac{1}{p}}(y)\|^{p'+p'\epsilon} dy \right)^{\frac{1}{p'+p'\epsilon}} \left(\frac{1}{|I|} \int_I |\vec{f}(y)|^{\frac{p+p\epsilon}{1+p\epsilon}} dy \right)^{\frac{1+p\epsilon}{p+p\epsilon}} \\
&\lesssim \|V_I A_I V_I^{-1}\| \left(\frac{1}{|I|} \int_I |\vec{f}(y)|^{\frac{p+p\epsilon}{1+p\epsilon}} dy \right)^{\frac{1+p\epsilon}{p+p\epsilon}}.
\end{aligned}$$

Thus, by Lemma 3.2, we are reduced to estimating

$$\int_{\mathbb{R}^d} \left(\sum_{I \in \mathcal{D}} \frac{\|V_I A_I V_I^{-1}\|^2}{|I|} \left(m_I |\vec{f}|^{\frac{p+p\epsilon}{1+p\epsilon}} \right)^{\frac{2+2p\epsilon}{p+p\epsilon}} \chi_I(t) \right)^{\frac{p}{2}} dt.$$

However, condition (b) precisely says that $\{\|V_I A_I V_I^{-1}\|\}_{I \in \mathcal{D}}$ is a Carleson sequence, so an application of Carleson's Lemma (Lemma 5.3 in [17]) gives us that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(\sum_{I \in \mathcal{D}} \frac{\|V_I A_I V_I^{-1}\|^2}{|I|} \left(m_I |\vec{f}|^{\frac{p+p\epsilon}{1+p\epsilon}} \right)^{\frac{2+2p\epsilon}{p+p\epsilon}} \chi_I(t) \right)^{\frac{p}{2}} dt \\
&\lesssim \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \sup_{\mathcal{D} \ni I \ni x} \left(m_I |\vec{f}|^{\frac{p+p\epsilon}{1+p\epsilon}} \right)^{\frac{2+2p\epsilon}{p+p\epsilon}} \frac{\chi_I(t)}{|I|} dx \right)^{\frac{p}{2}} dt \\
&\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \sup_{\mathcal{D} \ni I \ni t} \left(m_I |\vec{f}|^{\frac{p+p\epsilon}{1+p\epsilon}} \right)^{\frac{2+2p\epsilon}{p+p\epsilon}} \frac{\chi_I(x)}{|I|} dx \right)^{\frac{p}{2}} dt \\
&= \int_{\mathbb{R}^d} \left(M^d(|\vec{f}|^{\frac{p+p\epsilon}{1+p\epsilon}})(t) \right)^{\frac{p+2\epsilon}{p+p\epsilon}} dt \\
&\lesssim \int_{\mathbb{R}^d} |\vec{f}(t)|^p dt
\end{aligned}$$

where M^d is the ordinary dyadic maximal function, which completes the proof that (b) \Rightarrow (a).

(a) \Rightarrow (b): Fixing $J \in \mathcal{D}$, plugging in the test functions $\vec{f} := \chi_J \vec{e}_i$ into Π_A for any orthonormal basis \vec{e}_i of \mathbb{C}^n , and using (a) combined with dyadic Littlewood-Paley theory and elementary linear algebra gives us

that

$$\begin{aligned} |J| &\gtrsim \int_{\mathbb{R}^d} \left(\sum_{I \in \mathcal{D}} \frac{\|V_I A_I m_I(\chi_J W^{-\frac{1}{p}})\|^2}{|I|} \chi_I(x) \right)^{\frac{p}{2}} dx \\ &\geq \int_J \left(\sum_{I \in \mathcal{D}(J)} \frac{\|V_I A_I m_I(W^{-\frac{1}{p}})\|^2}{|I|} \chi_I(x) \right)^{\frac{p}{2}} dx \end{aligned}$$

which says that

$$\sup_{J \in \mathcal{D}} \frac{1}{|J|} \int_J \left(\sum_{I \in \mathcal{D}(J)} \frac{\|V_I A_I m_I(W^{-\frac{1}{p}})\|^2}{|I|} \chi_I(x) \right)^{\frac{p}{2}} dx < \infty.$$

Condition (b) now follows immediately from Lemma 3.2 and Theorem 3.1 in [15].

We now prove that (c) \Rightarrow (b) and (a) \Rightarrow (c) for the case $2 \leq p < \infty$.

(c) \Rightarrow (b) when $2 \leq p < \infty$: Note that condition (c) is equivalent to

$$\frac{1}{|K|} \sum_{I \in \mathcal{D}(K)} \|V_K^{-1} A_I^* V_I^2 A_I V_K^{-1}\| \lesssim 1$$

for any $K \in \mathcal{D}$. Fix $J \in \mathcal{D}$ and for each $j \in \mathbb{N}$ let $\mathcal{J}^j(J)$ and $\mathcal{F}^j(J)$ be defined as they were in Section 2 where $\lambda_1, \lambda_2 > 1$ are large enough so that Lemma 2.1 is true. Then inequality (2.1) tells us that

$$\begin{aligned} &\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \|V_I A_I V_I^{-1}\|^2 \\ &\leq \frac{1}{|J|} \sum_{j=1}^{\infty} \sum_{K \in \mathcal{J}^{j-1}(J)} \sum_{I \in \mathcal{F}^j(K)} \|V_I^{-1} V_K\| \|V_K^{-1} A_I^* V_I^2 A_I V_K^{-1}\| \|V_K V_I^{-1}\| \\ &\lesssim \|W\|_{\mathbb{A}_p}^{\frac{2}{p'}} \frac{1}{|J|} \sum_{j=1}^{\infty} \sum_{K \in \mathcal{J}^{j-1}(J)} \sum_{I \in \mathcal{D}(K)} \|V_K^{-1} A_I^* V_I^2 A_I V_K^{-1}\| \\ &\lesssim \|W\|_{\mathbb{A}_p}^{\frac{2}{p'}} \frac{1}{|J|} \sum_{j=1}^{\infty} \sum_{K \in \mathcal{J}^{j-1}(J)} |K| \\ &\lesssim \|W\|_{\mathbb{A}_p}^{\frac{2}{p'}} \sum_{j=1}^{\infty} 2^{-j} \lesssim \|W\|_{\mathbb{A}_p}^{\frac{2}{p'}}. \end{aligned}$$

(a) \Rightarrow (c) when $2 \leq p < \infty$: Fix $J \in \mathcal{D}$ and $\vec{e} \in \mathbb{C}^n$. If $\vec{f} = W^{\frac{1}{p}} \chi_J \vec{e}$, then condition (a), the definition of V_J , and Hölder's inequality give us

that

$$\begin{aligned}
|J| |V_J \vec{e}|^p &\gtrsim \int_{\mathbb{R}^d} \left(\sum_{I \in \mathcal{D}} \frac{|V_I A_I m_I(\chi_J) \vec{e}|^2}{|I|} \chi_I(t) \right)^{\frac{p}{2}} dt \\
&\geq |J| \left[\frac{1}{|J|} \int_J \left(\sum_{I \in \mathcal{D}(J)} \frac{|V_I A_I \vec{e}|^2}{|I|} \chi_I(t) \right)^{\frac{p}{2}} dt \right] \\
&\geq |J| \left[\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} |V_I A_I \vec{e}|^2 \right]^{\frac{p}{2}}
\end{aligned}$$

which proves (c), and in fact shows that (a) \Leftrightarrow (b) \Leftrightarrow (c) when $2 \leq p < \infty$. We will now complete the proof when $1 < p \leq 2$.

(b) \Rightarrow (c) when $1 < p \leq 2$: To avoid confusion in the subsequent arguments, we will write $V_I = V_I(W, p)$ to indicate which W and p the V_I at hand is referring to. As mentioned before, it is easy to see that W is a matrix A_p weight if and only if $W^{1-p'}$ is a matrix $A_{p'}$ weight. Furthermore, one can easily check that $V_I(W^{1-p'}, p') = V'_I(W, p)$ and $V'_I(W^{1-p'}, p') = V_I(W, p)$. Now if (b) is true, then the two equalities above give us that

$$\begin{aligned}
&\sup_{J \in \mathcal{D}} \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \|V_I(W^{1-p'}, p') A_I^* V'_I(W^{1-p'}, p')\|^2 \\
&= \sup_{J \in \mathcal{D}} \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \|V_I(W, p) A_I V'_I(W, p)\|^2 < \infty.
\end{aligned}$$

However, repeating word for word the proofs of (b) \Rightarrow (a) \Rightarrow (c) for the case $2 \leq p < \infty$ (where $W^{1-p'}$ replaces W and $A^* := \{A_I^*\}_{I \in \mathcal{D}}$ replaces the sequence A) gives us that there exists $C > 0$ where

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} A_I (V_I(W^{1-p'}, p'))^2 A_I^* < C (V_J(W^{1-p'}, p'))^2,$$

which proves (c) when $1 < p \leq 2$.

(c) \Rightarrow (b) when $1 < p \leq 2$: This follows immediately by again replacing W with $W^{1-p'}$, replacing A with $A^* := \{A_I^*\}_{I \in \mathcal{D}}$, and using the proof of (c) \Rightarrow (b) when $2 \leq p < \infty$.

Since (a) \Leftrightarrow (b) was shown for all $1 < p < \infty$, we therefore have that (a) \Leftrightarrow (b) \Leftrightarrow (c) for all $1 < p < \infty$. Finally, a careful checking of the

above arguments reveals that the $L^p(\mathbb{R}^d; \mathbb{C}^n)$ norm of Π_A is equivalent to the canonical supremums defined by conditions (b) and (c). \square

4. COUNTEREXAMPLES AND OPEN PROBLEMS

We begin this section with a proof of the following result that was mentioned in the introduction.

Proposition 4.1. *Let $1 < p < \infty$ and let W be a matrix A_p weight. If \mathcal{D} is any dyadic system of cubes and $A := \{A_I\}_{I \in \mathcal{D}}$ is a sequence of matrices, then the Haar multiplier*

$$T_A \vec{f} := \sum_{I \in \mathcal{D}} A_I \vec{f}_I h_I$$

is bounded on $L^p(W)$ if and only if $\sup_{I \in \mathcal{D}} \|V_I A_I V_I^{-1}\| < \infty$.

Proof. If $M = \sup_{I \in \mathcal{D}} \|V_I A_I V_I^{-1}\| < \infty$, then two applications of (1.4) give us that

$$\begin{aligned} \|T_A \vec{f}\|_{L^p(W)}^p &\approx \int_{\mathbb{R}^d} \left(\sum_{I \in \mathcal{D}} \frac{|V_I A_I \vec{f}_I|^2}{|I|} \chi_I(x) \right)^{\frac{p}{2}} dx \\ &\leq \int_{\mathbb{R}^d} \left(\sum_{I \in \mathcal{D}} \frac{\|V_I A_I V_I^{-1}\|^2 |V_I \vec{f}_I|^2}{|I|} \chi_I(x) \right)^{\frac{p}{2}} dx \\ &\leq M^p \int_{\mathbb{R}^d} \left(\sum_{I \in \mathcal{D}} \frac{|V_I \vec{f}_I|^2}{|I|} \chi_I(x) \right)^{\frac{p}{2}} dx \\ &\approx M^p \|\vec{f}\|_{L^p(W)}^p. \end{aligned}$$

For the other direction, fix some $J_0 \in \mathcal{D}$ and let $J'_0 \in \mathcal{D}(J_0)$ with $\ell(J'_0) = \frac{1}{2}\ell(J_0)$. Again by (1.4) we have that

$$\int_{\mathbb{R}^d} \left(\sum_{I \in \mathcal{D}} \frac{|V_I A_I (W^{-\frac{1}{p}} \vec{f})_I|^2}{|I|} \chi_I(x) \right)^{\frac{p}{2}} dx \lesssim \|\vec{f}\|_{L^p}^p. \quad (4.1)$$

Plugging $\vec{f} := \chi_{J'_0} \vec{e}$ for any $\vec{e} \in \mathbb{C}^n$ into (4.1) and noticing that

$$(W^{-\frac{1}{p}} \chi_{J'_0} \vec{e})_{J_0} = \pm 2^{-n} |J_0|^{\frac{1}{2}} m_{J'_0} (W^{-\frac{1}{p}})$$

gives us that $\|V_{J_0} A_{J_0} m_{J'_0} (W^{-\frac{1}{p}})\| \lesssim 1$. However, Lemma 3.2 then tells us that $\|V_{J_0} A_{J_0} V'_{J'_0}\| \lesssim 1$. Using the definition of $V_{J'_0}$ and summing over

all of the 2^n first generation children J'_0 of J_0 finally (after taking the supremum over $J_0 \in \mathcal{D}$) gives us that $\sup_{J \in \mathcal{D}} \|V_J A_J V'_J\| < \infty$, which implies that $\sup_{J \in \mathcal{D}} \|V_J A_J V_J^{-1}\| < \infty$ as desired. \square

We will now construct the examples mentioned in the introduction. First let

$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } W := \begin{pmatrix} |x|^\alpha & 0 \\ 0 & |x|^\beta \end{pmatrix}$$

for $x \in \mathbb{R}$, where $-1 < \pm\alpha, \pm\beta < p-1$ and $\alpha \neq \beta$ so that W is trivially a matrix A_p weight. (In fact, one can easily check that $V_I = (m_I W)^{\frac{1}{p}}$ and $V'_I = (m_I (W^{-\frac{p'}{p}}))^{\frac{1}{p'}}$ since W is diagonal). Now let T be the matrix kernelled CZO with kernel $K(x, y) = (x-y)^{-1}A$. Then obviously $T1 = T^*1 = 0$, but one can very easily check that T is not bounded on $L^p(W)$. On the other hand, let $\{A_I\}_{I \in \mathcal{D}}$ for any dyadic grid \mathcal{D} be the constant sequence $A_I = A$. Then one can easily check that

$$\sup_{I \in \mathcal{D}} \|V_I A V_I^{-1}\| \geq \sup_{I \in \mathcal{D}} \|V_I A V'_I\| = \infty$$

We will now show that $B \in \text{BMO}$ and W being a matrix A_p weight is not sufficient for π_B to be bounded on $L^p(W)$ for any $1 < p < \infty$, which will turn out to be more involved than the above examples. First we will need the following result, which is potentially of independent interest and whose proof is similar to the proof of Lemma 2.1 in [4].

Proposition 4.2. *Let Π_A be the operator defined in Theorem 1.3 for some fixed sequence of matrices $A := \{A_I\}_{I \in \mathcal{D}}$ and some fixed matrix A_p weight W . If Π_A is bounded on $L^p(\mathbb{R}^d; \mathbb{C}^n)$, then Π_A is bounded on $L^q(\mathbb{R}^d; \mathbb{C}^n)$ for all $p \leq q < \infty$.*

Proof. Since

$$\Pi_A^* \vec{f} = W^{-\frac{1}{p}} \sum_{I \in \mathcal{D}} \frac{A_I^* V_I \vec{f}_I \chi_I}{|I|},$$

we have that $\Pi_A^*(h_J \vec{e})$ is supported on J for each \vec{e} . One can then use the Calderón-Zygmund decomposition to check that Π_A^* is weak type $(1, 1)$, which by interpolation gives us that Π_A^* is bounded on $L^q(\mathbb{R}^d; \mathbb{C}^n)$ for all $1 < q < p'$. Duality now completes the proof. \square

Now let \mathcal{D} be any dyadic system of intervals in \mathbb{R} . For any $\alpha = \alpha(p) > 0$ with $-1 < \pm\alpha < p-1$, let W be the 2×2 matrix A_p weight

defined on \mathbb{R} by

$$W(x) := \begin{pmatrix} |x|^\alpha & 0 \\ 0 & |x|^{-\alpha} \end{pmatrix}.$$

Now let $b(x) := \log|x|$ and let $N \in \mathbb{N}$. Pick any $J_N \in \mathcal{D}$ such that $J_N \subseteq [2^{-N-1}, 2^{-N})$ and $|J_N| = 2^{-N-2}$. Now assume that $N \in \mathbb{N}$ is in fact large enough where

$$\frac{1}{|J_N|} \sum_{I \in \mathcal{D}(J_N)} |b_I|^2 > \frac{1}{2} \|b\|_{\text{BMO}}.$$

Let

$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B := bA,$$

and let $\vec{f}_N := \chi_{J_N} W^{-\frac{1}{p'}} A \vec{e}$ where

$$\vec{e} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

First we show that if $2 \leq q < \infty$, then $M_{W,p} \pi_B W^{-\frac{1}{p}}$ can not be bounded on $L^q(\mathbb{R}; \mathbb{C}^2)$. Note that

$$\begin{aligned} \|M_{W,p} \pi_B W^{-\frac{1}{p}} \vec{f}_N\|_{L^q}^q &\gtrsim \int_{J_N} \left(\sum_{I \in \mathcal{D}(J_N)} \frac{|V_I B_I m_I(W^{-1}) A \vec{e}|^2}{|I|} \chi_I(t) \right)^{\frac{q}{2}} dt \\ &= \int_{J_N} \left(\sum_{I \in \mathcal{D}(J_N)} \frac{|b_I|^2 |V_I m_I(W) \vec{e}|^2}{|I|} \chi_I(t) \right)^{\frac{q}{2}} dt \\ &\geq |J_N| \left[\frac{1}{|J_N|} \int_{J_N} \sum_{I \in \mathcal{D}(J_N)} \frac{|b_I|^2 |V_I m_I(W) \vec{e}|^2}{|I|} \chi_I(t) dt \right]^{\frac{q}{2}} \\ &\approx |J_N| \left[\frac{1}{|J_N|} \sum_{I \in \mathcal{D}(J_N)} |b_I|^2 \left(\frac{1}{|I|} \int_I |W^{\frac{1}{p}}(t) m_I(W) \vec{e}|^p dt \right)^{\frac{2}{p}} \right]^{\frac{q}{2}} \\ &\gtrsim 2^{-N} \left[\frac{\|b\|_{\text{BMO}}}{2} 2^{(2\alpha + \frac{2\alpha}{p})N} \right]^{\frac{q}{2}}. \end{aligned} \tag{4.2}$$

However,

$$\begin{aligned}
\|\vec{f}_N\|_{L^q}^q &= \int_{J_N} |W^{-\frac{1}{p'}}(t) A \vec{e}|^q dt \\
&\approx 2^{-N} \left[2^{\frac{2\alpha}{p'} N} \right]^{\frac{q}{2}} \\
&= 2^{-N} \left[2^{(2\alpha - \frac{2\alpha}{p}) N} \right]^{\frac{q}{2}}
\end{aligned} \tag{4.3}$$

which shows that $M_{W,p}\pi_B W^{-\frac{1}{p}}$ can not be bounded on $L^q(\mathbb{R}; \mathbb{C}^2)$.

This trivially means that $M_{W,p}\pi_B W^{-\frac{1}{p}}$ can not be bounded on $L^p(\mathbb{R}; \mathbb{C}^2)$ if $2 \leq p < \infty$. However, if $1 < p < 2$ and $M_{W,p}\pi_B W^{-\frac{1}{p}}$ is bounded on $L^p(\mathbb{R}; \mathbb{C}^2)$, then Proposition 4.1 says that $M_{W,p}\pi_B W^{-\frac{1}{p}}$ is bounded on $L^q(\mathbb{R}; \mathbb{C}^2)$ for all $p \leq q < \infty$, which is a contradiction. Thus, for any $1 < p < \infty$, $M_{W,p}\pi_B W^{-\frac{1}{p}}$ can not be bounded on $L^p(\mathbb{R}; \mathbb{C}^2)$, or equivalently, π_B is not bounded on $L^p(W)$. \square

We will end this paper with a discussion of some open problems related to this paper. First, note that Theorem 1.3 is unconditional in the following sense. Suppose that we are given a sequence $A := \{A_I\}_{I \in \mathcal{D}}$ and a matrix A_p weight W where A and W together satisfy any of the conditions in Theorem 1.3. Now let $\sigma := \{\sigma_I\}_{I \in \mathcal{D}} \in \{-1, 1\}^{\mathcal{D}}$ be arbitrary and let $A_\sigma := \{\sigma_I A_I\}_{I \in \mathcal{D}}$. Then it is immediate from condition (b) in Theorem 1.3 that A_σ and W together satisfy any of the conditions in Theorem 1.3. Furthermore, Theorem 1.3 says that the $L^p(\mathbb{R}^d; \mathbb{C}^n)$ operator norm of Π_{A_σ} is equivalent to the canonical supremums defined by conditions (b) and (c) which are both trivially independent of σ . Moreover, if one replaces A_I by $V_I^{-1} A_I$ then (c) becomes

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} A_I^* A_I < C V_J$$

which is trivially unconditional in the sense that one can even replace A_I by $U_I A_I$ for any sequence $\{U_I\}_{I \in \mathcal{D}}$ of unitary matrices without (modulo constants) modifying the operator norm of (the appropriately modified version of) Π_A . It would be interesting to know if it is possible to exploit this unconditionality to somehow improve Theorem 1.3 or to obtain some new results.

Now note that one might ask whether BMO_W^p when W is a matrix A_p weight for $1 < p < \infty$ coincides with the space of locally integrable

functions $B : \mathbb{R}^d \rightarrow \mathbb{C}^n$ where

$$\sup_{\substack{I \subset \mathbb{R}^d \\ I \text{ is a cube}}} \frac{1}{|I|} \int_I \|V_I(B(x) - m_I B)V_I^{-1}\|^2 dx < \infty. \quad (4.4)$$

Interestingly, one can prove the following “matrix weighted John-Nirenberg theorem” relatively easily using our stopping time and Theorem 3.1 in [15]. It should be noted that this John-Nirenberg type result will play a crucial role in the proof of sufficiency in Theorem 1.1.

Lemma 4.3. *Let $1 < p, q < \infty$ and suppose that W is a matrix A_p weight. If $BMO_W^{p,q}$ is the space of matrix functions B such that*

$$\sup_{\substack{I \subset \mathbb{R}^d \\ I \text{ is a cube}}} \frac{1}{|I|} \int_I \|V_I(B(x) - m_I B)V_I^{-1}\|^q dx < \infty,$$

then we have that

$$\bigcup_{1 < q < \infty} BMO_W^{p,q} \subseteq BMO_W^p.$$

Proof. Let $B \in BMO_W^{p,q}$ for some $q > 1$ so by dyadic Littlewood-Paley theory,

$$\sup_{\substack{I \subset \mathbb{R}^d \\ I \text{ is a cube}}} \frac{1}{|I|} \int_I \left(\sum_{J \in \mathcal{D}(I)} \frac{\|V_I B_J V_I^{-1}\|^2}{|J|} \chi_J(x) \right)^{\frac{q}{2}} dx < \infty. \quad (4.5)$$

However, by the definition of BMO_W^p and Theorem 3.1 in [15], we need to prove

$$\sup_{\substack{I \subset \mathbb{R}^d \\ I \text{ is a cube}}} \frac{1}{|I|} \int_I \left(\sum_{J \in \mathcal{D}(I)} \frac{\|V_J B_J V_J^{-1}\|^2}{|J|} \chi_J(x) \right)^{\frac{q}{2}} dx < \infty. \quad (4.6)$$

Clearly by Hölder’s inequality we can assume that $1 < q \leq 2$. Note that $J \in \mathcal{F}(K)$ implies that $\|V_J V_K^{-1}\| \lesssim 1$ and $\|V_K V_J^{-1}\| \lesssim \|W\|_{A_p}^{\frac{1}{p'}}$, so that for fixed $I \in \mathcal{D}$,

$$\begin{aligned}
& \frac{1}{|I|} \int_I \left(\sum_{J \in \mathcal{D}(I)} \frac{\|V_J B_J V_J^{-1}\|^2}{|J|} \chi_J(x) \right)^{\frac{q}{2}} dx \\
&= \frac{1}{|I|} \int_I \left(\sum_{j=1}^{\infty} \sum_{K \in \mathcal{J}^{j-1}(I)} \sum_{J \in \mathcal{F}(K)} \frac{\|V_J B_J V_J^{-1}\|^2}{|J|} \chi_J(x) \right)^{\frac{q}{2}} dx, \\
&\lesssim \frac{\|W\|_{A_p}^{\frac{q}{p'}}}{|I|} \int_I \left(\sum_{j=1}^{\infty} \sum_{K \in \mathcal{J}^{j-1}(I)} \sum_{J \in \mathcal{D}(K)} \frac{\|V_K B_J V_K^{-1}\|^2}{|J|} \chi_J(x) \right)^{\frac{q}{2}} dx \\
&\leq \frac{\|W\|_{A_p}^{\frac{q}{p'}}}{|I|} \sum_{j=1}^{\infty} \sum_{K \in \mathcal{J}^{j-1}(I)} \int_K \left(\sum_{J \in \mathcal{D}(K)} \frac{\|V_K B_J V_K^{-1}\|^2}{|J|} \chi_J(x) \right)^{\frac{q}{2}} dx. \quad (4.7)
\end{aligned}$$

However by (4.5) we have that

$$(4.7) \lesssim \frac{\|W\|_{A_p}^{\frac{q}{p'}}}{|I|} \sum_{j=1}^{\infty} \sum_{K \in \mathcal{J}^{j-1}(I)} |K| \lesssim \|W\|_{A_p}^{\frac{q}{p'}} \sum_{j=1}^{\infty} 2^{-(j-1)} < \infty$$

which proves (4.6). \square

Unfortunately, it is not clear at all whether there even exists a $q > 1$ depending on p and W where $\text{BMO}_W^p \subseteq \text{BMO}_W^{p,q}$.

Furthermore, we do not know whether it is in fact necessary to separate the cases $1 < p \leq 2$ and $2 \leq p < \infty$ in the original definition of BMO_W^p (and in condition (c) in Theorem 1.3) when W is a matrix A_p weight. In particular, recall that if w is a scalar A_∞ weight, then a scalar function b satisfies (1.3) if and only if $b \in \text{BMO}$. Using this fact and the basic properties of A_∞ weights in conjunction with the classical John-Nirenberg theorem, it is easy to see that if w is a scalar A_p weight, then b satisfying either of the separate conditions defining BMO_W^p (or equivalently, either of the conditions in (c) of Theorem 1.3) regardless of the range of p is true if and only if $b \in \text{BMO}$. In other words, if separating the cases $1 < p \leq 2$ and $2 \leq p < \infty$ is in fact necessary, then this is exclusively a *noncommutative* phenomenon that is *not* present in the scalar setting, and may very well be due to a lack of one single “matrix A_∞ class.” (In fact, one instead has a family of classes $A_{p,\infty}$ for each $1 < p < \infty$ which all coincide with A_∞ in the

scalar setting.) Note that we will investigate both these matters more carefully in part II.

We will end this paper with a brief discussion of quantitative weighted norm inequalities. Note that since $|W^{\frac{1}{p}}(x)\vec{e}|^p$ is a scalar A_p weight with characteristic less than $\|W\|_{A_p}$ for each fixed \vec{e} , we can in fact let $q \approx p + \|W\|_{A_p}$ and $q \approx p' + \|W\|_{A_p}^{p-1}$ in our uses of the reverse Hölder inequality.

Thus, one can carefully analyse our proofs to obtain concrete dependence of our results on the A_p characteristic. Of particular interest is the case when $p = 2$, and unfortunately our results are most likely not sharp (with the exception of (b) \Rightarrow (a) in Theorem 1.3, which in particular says that $\|\Pi_A\|_{L^2 \rightarrow L^2} \lesssim \|W\|_{A_2} \|A\|_{\text{Car}}$ where $\|\cdot\|_{\text{Car}}$ is the canonical norm associated to a Carleson sequence.)

Moreover, recall that the implication (c) \Rightarrow (a) in Theorem 1.3 when $p = 2$ gives us that

$$\sum_{I \in \mathcal{D}} |A_I m_I \vec{f}|^2 \lesssim C \|\vec{f}\|_{L^2(W)}^2 \quad (4.8)$$

whenever W is a matrix A_2 weight and $\{A_I\}_{I \in \mathcal{D}}$ is a “ W -Carleson sequence” of matrices in the sense that there exists $C > 0$ independent of $J \in \mathcal{D}$ such that

$$\sum_{I \in \mathcal{D}(J)} A_I^* A_I < C \int_J W(t) dt.$$

Also, recall that this was implicitly used in sharp form by O. Beznosova in [2] to prove sharp weighted norm inequalities for scalar paraproducts. Clearly for the sake of proving sharp matrix weighted norm inequalities for scalar kernelled CZOs (and in particular, the Hilbert transform) or other related operators (such as scalar symbolled dyadic paraproducts or the martingale transform) it would be important to prove (4.8) but with sharp dependence on the matrix A_2 characteristic.

Unfortunately, it is clear that our approach does not provide sharp dependence of (4.8) on the matrix A_2 condition, and in particular we have proved that $C \lesssim \|W\|_{A_2}^{\frac{3}{2}}$ where C is from (4.8). Furthermore, our approach requires one to prove (c) \Rightarrow (a) in Theorem 1.3 by first proving (c) \Rightarrow (b) with our stopping time, which adds in a $\|W\|_{A_2}^{\frac{1}{2}}$ term and is most likely unnecessary. This immediately raises the question of whether one can *directly* prove that (c) \Rightarrow (a) in Theorem 1.3 when

$p = 2$. Furthermore, the fact that our stopping time approach even works to prove Theorems 1.3 and 2.3 raises the possibility that one could extend the more sophisticated stopping time from [12] to prove sharper versions of our results.

Finally, if w is *any* scalar weight and if

$$m_{I,w}f := w(I)^{-1} \int_I f(x)w(x) dx,$$

then recall that “the” weighted Carleson embedding theorem in the scalar $p = 2$ setting states that

$$\sum_{I \in \mathcal{D}} |a_I^2(m_{I,w}f)|^2 \lesssim C \|\vec{f}\|_{L^2(w)}^2$$

whenever $\{a_I\}_{I \in \mathcal{D}}$ is a scalar w -Carleson sequence. By arguments very similar to the proof of Theorem 1.3, one can show that

$$\sum_{I \in \mathcal{D}} |(m_I W)^{-\frac{1}{2}} A_I^* (m_I W)^{-\frac{1}{2}} m_I(W \vec{f})|^2 \lesssim \|W\|_{A_2}^2 \|\vec{f}\|_{L^2(W)}^2 \quad (4.9)$$

whenever W is a matrix A_2 weight and $\{A_I\}_{I \in \mathcal{D}}$ is a W -Carleson sequence of matrices.

Clearly the fact that (4.9) requires W to be a matrix A_2 weight leaves much to be desired from the point of view of sharp matrix weighted norm inequalities or two matrix weighted norm inequalities, and it would therefore be very interesting to prove (4.9) for general matrix weights. Note that this will probably be a very challenging task, especially considering the fact that it is not clear at all if there even exists a matrix weighted maximal function that is bounded on $L^2(W)$ *without* requiring W to be a matrix A_2 weight. Moreover, even if one were to produce such a matrix maximal function, it is not immediate how one would put such a maximal function to use (see [8] for more on matrix weighted maximal functions.)

Finally, let H be the classical Hilbert transform on \mathbb{R} . Then in the very recent preprint [3], the authors have established the estimate

$$\|H\|_{L^2(W) \rightarrow L^2(W)} \lesssim \|W\|_{A_2}^{\frac{3}{2}} \log \|W\|_{A_2} \quad (4.10)$$

by utilizing the arguments in [18] in conjunction with a sharper version of Theorem 2.3 when $p = 2$ (in terms of A_2 characteristic dependence of the operator norm). Unfortunately, because of the difficulties mentioned above, it seems like improving (4.10) by removing even the $\log \|W\|_{A_2}$ term will be a highly nontrivial task.

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